

ON LIMITS OF SCALAR OPERATORS

BY

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1. Introduction. The basic problem under consideration in the first part of this paper is that of determining conditions under which a weak or strong limit of scalar operators is again a scalar operator. This question has been treated by Bade [1], Foguel [7], [8], and Edwards and Ionescu Tulcea [6], among others. The central result considered here is a generalization of a theorem due to Rellich [12]. Foguel's approach is used below to obtain a variation of the previous forms of Rellich's theorem which is valid in locally convex spaces which are barreled, separated, and quasi-complete. This is the Corollary of Theorem 3 below. In the last section the connection between this theorem and Foguel's is explored. There is also included some material on Boolean algebras (B.A.'s) of projections, which generalizes many of the theorems of Bade in [1] and [2].

In the process of this work we give a generalization to locally convex spaces of a theorem of Dunford's to the effect that a bounded Boolean algebra of projections on a weakly complete space is contained in a σ -complete B.A. (which is in turn contained in the strong closure of the original B.A.). As noted in [2], the proof given in [4, p. 578], for a special case, does indeed generalize, even to the locally convex case. The proof given below, however, is somewhat different, although the essential tool, namely the family of measures constructed at the beginning of the proof, is the same.

2. Preliminary concepts. Throughout this paper E will denote a locally convex linear topological space over the complex field C , which is separated, barreled, and quasi-complete. We denote by E' the strong dual of E , that is, the space of linear continuous mappings of E into the complex field, endowed with the topology of uniform convergence on bounded subset of E . Let \mathcal{C} be the family of equicontinuous subsets of E' . If F is any linear topological space, $L(E, F)$ will denote the space of continuous linear mappings of E into F . The dual map of an element U of $L(E, F)$ will be denoted by $'U$. For Z a locally compact space, we denote by $\mathcal{K}(Z)$ the algebra of all complex-valued continuous functions defined on Z which have compact support, and by $M^1(Z)$ the Banach space of

Received by the editors July 6, 1965.

(1) Part of the contents of this paper are taken from the author's Ph. D. dissertation, Yale University. The remainder was supported by National Science Foundation research contract GP-1233. Some of the contents were presented to the Society, April 9, 1965.

all complex bounded Radon measures on Z . The characteristic function of a subset D of Z will be denoted ϕ_D . Let $B_0(Z)$ denote the algebra of complex-valued Baire functions, $B_0^\infty(Z)$ the bounded functions in $B_0(Z)$, and $S_0(Z)$ the Baire sets of Z . The symbols $B(Z)$, $B^\infty(Z)$, and $S(Z)$ are similarly defined, using "Borel" instead of "Baire." A family $\mathcal{F} = \{m_{x,x'}\}_{x \in E, x' \in E'}$ of bounded Radon measures on Z is said to be semispectral if the mapping $h: (x, x') \rightarrow m_{x,x'}$ of $E \times E' \rightarrow M^1(Z)$ is bilinear and \mathcal{E} -hypocontinuous. \mathcal{E} -hypocontinuous means that h is separately continuous and for every neighborhood W of zero in $M^1(Z)$ and every set M in \mathcal{E} there is a neighborhood V of zero in E such that $h(V \times M) \subseteq W$. We denote by $L^\infty(\mathcal{F})$ the set of all complex-valued functions on Z which are $m_{x,x'}$ -measurable for every $x \in E$ and $x' \in E'$, for which $N^\infty(f, \mathcal{F}) = \inf\{\alpha > 0 : \{z : |f(z)| > \alpha\} \text{ is } m_{x,x'}\text{-negligible for all } x \in E, x' \in E'\}$ is finite. For fixed $x \in E, f \in L^\infty(\mathcal{F})$, define a mapping $U_f x$ of E' into C by $\langle U_f x, x' \rangle = \int_Z f dm_{x,x'}$. It follows, by reasonably straightforward calculations, using the properties of h , that the mapping $U_f: x \rightarrow U_f x$ has range in E'' , is linear, and is continuous when E'' has the topology of uniform convergence on elements of \mathcal{E} . Also, $\{U_f: N^\infty(f, \mathcal{F}) \leq 1\}$ is an equicontinuous subset of $L(E, E'')$. For if $A \in \mathcal{E}$ and $\varepsilon > 0$, there is a neighborhood V of zero in E such that $\|m_{x,x'}\| \leq \varepsilon$ for all $x \in V, x' \in A$. We deduce $|\langle U_f x, x' \rangle| \leq \varepsilon$ if $f \in L^\infty(\mathcal{F}), N^\infty(f, \mathcal{F}) \leq 1, x \in V$, and $x' \in A$. Since the polars of the sets of \mathcal{E} are a fundamental system of neighborhoods of zero in E'' , the equicontinuity just asserted is proved. We shall assume that for all our semispectral families, the range of U_f is in E for all $f \in L^\infty(\mathcal{F})$, and that $U_1 = I$, where 1 is the function $1(z) = 1$ for all $z \in Z$. A semispectral family of bounded Radon measures is said to be spectral if $f \cdot m_{x,x'} = m_{U_f x, x'} = m_{x, y'}$ where $y' = U_{f,x'} x'$ for every $f \in L^\infty(\mathcal{F}), x \in E$, and $x' \in E'$. If \mathcal{F} is spectral then the mapping $f \rightarrow U_{\mathcal{F}, f}$ is a representation of the algebra $L^\infty(\mathcal{F})$ into the algebra $L(E, E)$. If \mathcal{F} is semispectral and $f \cdot m_{x,x'} = m_{U_{\mathcal{F}, f} x, x'}$ for every $f \in B_0(Z), x \in E$, and $x' \in E'$, then \mathcal{F} is spectral. The above material is taken essentially from [10].

Let $\mathcal{F} = \{m_{x,x'}\}_{x \in E, x' \in E'}$ be a spectral family on Z . Define $P: S_0(Z) \rightarrow L(E, E)$ by the formula $P(D) = U_{\mathcal{F}, \phi_D}$. Then P has the properties

- (1) $D \rightarrow P(D)x$ is countably additive on $S_0(Z)$ for each $x \in E$;
- (2) $P(D' \cap D'') = P(D')P(D'')$ for all D', D'' in $S_0(Z)$;
- (3) $\{P(D) : D \in S_0(Z)\}$ is an equicontinuous subset of $L(E, E)$;
- (4) $P(Z) = I$.

Any mapping from $S_0(Z) \rightarrow L(E, E)$ with these four properties is called a *spectral measure* on Z . It can be shown that for every spectral measure P on Z , there is a unique spectral family \mathcal{F} on Z such that $P(D) = U_{\mathcal{F}, \phi_D}$ for every D in $S_0(Z)$. When \mathcal{F} is a spectral family on Z , it will often be convenient to write $\int_Z f dP$ for $U_{\mathcal{F}, f}$.

Let $\mathcal{F} = \{m_{x,x'}\}_{x \in E, x' \in E'}$ be a spectral family on Z , U an open subset of Z which contains the support of every measure in \mathcal{F} , and f a mapping of U into a locally compact space Y which is $m_{x,x'}$ -measurable for every $m_{x,x'}$ in \mathcal{F} . For each $x \in E$, $x' \in E'$, define $m_{x,x'}^f = f \cdot (m_{x,x'} \text{ restricted to } U)$. Then $\mathcal{F}^f = \{m_{x,x'}^f\}_{x \in E, x' \in E'}$ is a spectral family on Y and $U_{\mathcal{F}^f, g} = U_{\mathcal{F}, (g \circ f)}$, for all $g \in B^\infty(Y)$, where $(g \circ f)'(z) = g(f(x))$ if $z \in U$ and $= 0$ if $z \notin U$. \mathcal{F}^f is called the image of \mathcal{F} by f .

We shall say that an element T of $L(E, E)$ commutes with \mathcal{F} if $TU_{\mathcal{F}, f} = U_{\mathcal{F}, f}T$ for every $f \in B^\infty(Z)$. An operator $T \in L(E, E)$ is said to be a spectral operator if there is a spectral family \mathcal{F}_T on C such that

$$(5) \quad T \text{ commutes with } \mathcal{F}_T;$$

$$(6) \quad \text{sp}(T_\sigma) \subseteq \sigma \text{ for every compact } \sigma \subset C.$$

(For each compact σ , we denote by T_σ the mapping $x \rightarrow Tx$ of $E_\sigma \rightarrow E_\sigma$, where $E_\sigma = P(\sigma)E$.)

It can be shown that the spectral family \mathcal{F}_T associated with a spectral operator T is unique. The spectral measure $P_{\mathcal{F}_T}$ is sometimes called a resolution of the identity for T .

We say that an operator $S \in L(E, E)$ is scalar if there is a spectral family $\mathcal{F} = \{m_{x,x'}\}_{x \in E, x' \in E'}$ on C of measures with compact support such that $\int_C z dm_{x,x'}(z) = \langle Sx, x' \rangle$ for every $x \in E$, $x' \in E'$. In this case, we write $S = \int_C z dP_{\mathcal{F}}(z)$. See [11] concerning the above concepts and definitions.

3. The perturbation theorem. The basic perturbation theorem is given first, followed by some consequences which are more evidently generalizations of the theorems of Bade and Foguel.

THEOREM 1. Let P be a spectral measure on a locally compact space Z with associated spectral family $\mathcal{F} = \{m_{x,x'}\}_{x \in E, x' \in E'}$. Let $\{f_n\}$ be a sequence of functions in $B(Z)$, and f a complex-valued function defined on Z , such that f_n converges to f in $m_{x,x'}$ -measure for every $x \in E$, $x' \in E'$. Let $g \in B^\infty(C)$ be such that its discontinuities are contained in a closed set α . If $x_0 \in E$ has the property that $P(f^{-1}(\alpha))x_0 = 0$, then

$$(7) \quad \lim_{n \rightarrow \infty} \left(\int_Z g \cdot f_n dP \right) x_0 = \left(\int_Z g \cdot f dP \right) x_0.$$

Proof. Using a theorem of Grothendieck ([9, Theorem 2, p. 146], see also [6, p. 550–551]), it is possible, for every x_0 in E and every equicontinuous set A in E' , to find $x'_0 \in E'$ such that m_{x_0, x'_0} is positive and the measures $m_{x_0, x'}$, $x' \in A$, are uniformly absolutely continuous with respect to m_{x_0, x'_0} . This means that for each $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that $m_{x_0, x'_0}(\sigma) < \delta_\varepsilon$ implies $m_{x_0, x'}(\sigma) < \varepsilon$ for every $x' \in A$.

Let $G_n = \int_Z g \cdot f_n dP$ and $G = \int_Z g \cdot f dP$. Then for each $x \in E$, $x' \in E'$, $\varepsilon > 0$,

$$(8) \quad \begin{aligned} |\langle (G_n - G)x, x' \rangle| &= \left| \int_Z (g \cdot f_n - g \cdot f) dm_{x, x'} \right| \\ &\leq \int_{Z_{\varepsilon, n}} |g \cdot f_n - g \cdot f| d|m_{x, x'}| + \int_{Z_{\varepsilon, n}^c} |g \cdot f_n - g \cdot f| d|m_{x, x'}| \end{aligned}$$

where

$$Z_{\varepsilon, n} = \{z : |(g \cdot f_n)(z) - (g \cdot f)(z)| \geq \varepsilon\}.$$

We now fix $\varepsilon > 0$, A an equicontinuous set in E' , and let x_0 be as described in the statement of the theorem. Using Grothendieck's work to find x'_0 , we have by assumption that $f_n \rightarrow f$ in $m_{x_0, x'}$ -measure. Hence there is a set σ_0 in Z and a subsequence $\{f_{n_i}\} \subseteq \{f_n\}$ such that $m_{x_0, x'_0}(\sigma_0) = 0$ and $f_{n_i}(z) \rightarrow f(z)$ for all $z \notin \sigma_0$. By the continuity of g , $(g \cdot f_{n_i})(z) \rightarrow (g \cdot f)(z)$ for all $z \notin \sigma_0 \cup f^{-1}(x)$. Since $P(f^{-1}(x))x = 0$, we have $m_{x_0, x'_0}(\sigma_0 \cup f^{-1}(x)) = 0$. Hence we may choose N large enough so that $m_{x_0, x'_0}(Z_{\varepsilon, n_i}) \leq \delta_\varepsilon$ for all $n_i \geq N$. Combining this with (8) above, letting K be the bound of g , we get

$$\begin{aligned} |\langle (G_{n_i} - G)x_0, x' \rangle| &\leq 2K |m_{x_0, x'}|(Z_{\varepsilon, n_i}) + \varepsilon |m_{x_0, x'}|(Z) \\ &\leq \varepsilon(2K + L) \end{aligned}$$

for all $x' \in A$, $n_i \geq N$. Here we have let $L = \sup_{x' \in A} |m_{x_0, x'}|(Z)$. Hence $\langle G_{n_i}x_0, x' \rangle \rightarrow \langle Gx_0, x' \rangle$ uniformly for $x' \in A$. The same argument may be applied to any subsequence so that $\langle G_nx_0, x' \rangle \rightarrow \langle Gx_0, x' \rangle$ uniformly for $x' \in A$. Since A is arbitrary and polars of sets in \mathcal{E} generate the topology of E , $G_nx_0 \rightarrow Gx_0$ and (7) is proved.

THEOREM 2. Let P_1 be a spectral measure on a locally compact space Z , and P_2 a spectral measure on C . Let $\mathcal{F}_1 = \{m_{x, x'}^1\}_{x \in E, x' \in E'}$ and $\mathcal{F}_2 = \{m_{x, x'}^2\}_{x \in E, x' \in E'}$ be their respective spectral families of measures on Z and C . We suppose also that each $m_{x, x'}^2$ has compact support. If the range of P_1 contains the range of P_2 , then there is a function f in $B_0(Z)$ such that $\mathcal{F}_1^f = \mathcal{F}_2$.

Proof. For each positive integer n , let $\pi_n = \{\sigma_i^n, \lambda_i^{(n)}\}$ be a partition of $\text{Supp } \mathcal{F}_2$, of norm $|\pi_n| \leq 1/n$, with sets of $S_0(C)$. That is, for each n ,

$$(9) \quad \sigma_i^n \cap \sigma_j^n = \emptyset, \quad i \neq j,$$

$$(10) \quad \bigcup_{i=1}^{\infty} \sigma_i^n = \text{Supp } \mathcal{F}_2,$$

$$(11) \quad \text{diameter } \sigma_i^n \leq 1/n \quad \text{for every } i,$$

(12) every $\sigma_i^n \in S_0(C)$, and

(13) $\lambda_i^{(n)} \in \sigma_i^n$ for every i .

We can and do suppose further that $m > n$ implies that π_m is finer than π_n . Hence

(14) for each $m, n, m > n$, given any σ_i^m there
is a (unique) set $\sigma_{j_i}^n$ such that $\sigma_i^m \subseteq \sigma_{j_i}^n$.

Since both P_1 and P_2 are spectral measures, it is easy to verify (for arbitrary $B, B' \in S_0(Z)$ and $\sigma, \sigma' \in S_0(C)$) that

(15) $P_1(B) = P_2(\sigma)$ if and only if $P_1(\mathbf{C}B) = P_2(\mathbf{C}\sigma)$,

and

(16) if $\sigma \subset \sigma'$, $P_1(B) = P_2(\sigma)$, and $P_1(B') = P_2(\sigma')$,
then $P_1(B \cap B') = P_2(\sigma)$.

The range of P_2 being contained in the range of P_1 , we may associate with each σ_i^n a set $B_i^n \in S_0(Z)$ such that $P_1(B_i^n) = P_2(\sigma_i^n)$ for all i and n . From (14) and (16) we may choose the B_i^n so that for any $m, n, m > n$, we have

(17) for each B_i^m there is a (unique) set $B_{j_i}^n$ such that
 $B_i^m \subset B_{j_i}^n$. Moreover, $B_i^m \subset B_j^n$ if and only if $\sigma_i^m \subset \sigma_j^n$.

We now disjoint the sequences $\{B_i^n\}$ by defining

$$D_1^n = B_1^n \text{ and } D_i^n = B_i^n \cap \mathbf{C} \bigcup_{j=1}^{i-1} D_j^n, \quad i = 2, 3, \dots,$$

for each $n = 1, 2, \dots$. Then $D_i^n \cap D_j^n = \emptyset$ for $i \neq j$. A simple inductive argument, using (15), shows that $P_1(D_i^n) = P_2(\sigma_i^n)$ for all i and n . Also, the sets D_i^n enjoy the same property (17) that the B_i^n have. Finally, $P_1(\bigcup_{i=1}^{\infty} D_i^n) = I$ so that if $D_0 = \mathbf{C} \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} D_i^n$, then $P_1(D_0) = 0$.

We now define

$$f_n(z) = \sum_{i=1}^{\infty} \lambda_i^{(n)} \phi_{D_i^n}(z).$$

Then $\lim_n f_n(z) = 0$ for all $z \in D_0$. For $z \notin D_0$, $m > n$, it follows from (17) that $z \in D_i^m \cap D_j^n$ for just one i and one $j = j_i$. Hence, by (11) above,

$$|f_m(z) - f_n(z)| = |\lambda_i^{(m)} - \lambda_j^{(n)}| \leq 1/n,$$

independent of $z \in \mathbf{C}D_0$. Thus $\{f_n(z)\}$ is a uniformly Cauchy sequence and converges uniformly on $\mathbf{C}D_0$ to a function $f \in B_0(Z)$.

Using Theorem 1, for every $g \in \mathcal{K}(C)$, and every $x \in E$, we have

$$(18) \quad \lim_n \left(\int_Z g \cdot f_n dP \right) x = \left(\int_Z g \cdot f dP \right) x.$$

We now define $g_n(\lambda) = \sum_{i=1}^{\infty} g(\lambda_i^{(n)}) \phi_{\sigma^{n_i}}(\lambda)$. Then

$$(19) \quad \int_C g_n dP_2 = \int_Z g \cdot f_n dP_1 \quad \text{for each } n.$$

Moreover, $g_n \rightarrow g$ uniformly on $\text{Supp } m_{x,x'}^2$ for each $x \in E, x' \in E'$. Hence,

$$(20) \quad \lim_n \int_C g_n dm_{x,x'}^2 = \int_C g dm_{x,x'}^2.$$

Combining (18), (19), and (20), we have

$$(21) \quad \int_Z g \cdot f dm_{x,x'}^1 = \int_C g dm_{x,x'}^2 \quad \text{for all } g \in \mathcal{K}(C).$$

But this is all we need.

THEOREM 3. *Let $\{S_n\}$ be a sequence of scalar operators which converge strongly to a scalar operator S_0 . Let $\mathcal{F}_n = \{m_{x,x'}^n\}_{x \in E, x' \in E'}$ be the corresponding spectral families of measures on C , and P_n the corresponding spectral measures, $n = 0, 1, 2, \dots$. Suppose there is a spectral measure P on a locally compact space Z , with spectral family $\mathcal{F} = \{m_{x,x'}^P\}_{x \in E, x' \in E'}$, such that the range of P contains the range of every P_n . Let $g \in B_0^\infty(C)$ have its discontinuities in the closed set α . Then there are functions f_n in $B_0(Z)$ such that*

$$(22) \quad \langle S_n x, x' \rangle = \int_Z f_n dm_{x,x'}^P \quad \text{for } n = 0, 1, 2, \dots$$

and

$$(23) \quad f_n \rightarrow f_0 \text{ in } m_{x,x'}^P\text{-measure for every } x \in E, x' \in E'.$$

Moreover,

$$(24) \quad \text{if, for some } x_0 \in E, P_0(\alpha)x_0 = 0, \text{ then } g(S_n)x_0 \rightarrow g(S_0)x_0;$$

$$(25) \quad \text{if, for some } x_0 \in E \text{ and some } \beta \in S(C), \text{ we have}$$

$$P_0(\text{boundary } \beta)x_0 = 0, \text{ then } P_n(\beta)x_0 \rightarrow P_0(\beta)x_0;$$

and finally,

$$(26) \quad \begin{aligned} &\text{if } P_0(\alpha) = 0, \text{ then } g(S_0) \text{ is the strong limit of} \\ &g(S_n). \text{ (By } g(S_n), \text{ of course, is meant } \int_C g dP_n.) \end{aligned}$$

Proof. Statement (22) is an immediate result of the previous theorem.

For (23), let $B = \{g \in \mathcal{K}(Z) : \sup_{z \in Z} |g(z)| \leq 1\}$. Then, for each $x \in E$, $x' \in E'$,

$$\begin{aligned} \int_Z |f_n - f| d|m_{x,x'}^P| &= \sup_{g \in B} \left| \int_Z g(f_n - f) dm_{x,x'}^P \right| \\ &= \sup_{g \in B} \left| \int_Z (f_n - f) dm_{x, {}^tU_{\mathcal{F},g}x'}^P \right| \\ &= \sup_{g \in B} |\langle (S_n - S_0)x, {}^tU_{\mathcal{F},g}x' \rangle| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, because $S_n \rightarrow S_0$ strongly and $\{{}^tU_{\mathcal{F},g}x' : g \in B\}$ is an equicontinuous set for each $x' \in E'$.

Statements (24) and (26) follow from (23) and Theorem 1.

Statement (25) follows from (24) by letting $g = \phi_\beta$.

COROLLARY. Let S_n be a sequence of scalar operators, and P_n the corresponding spectral measures on C , $n = 1, 2, \dots$. Suppose there is a spectral measure P on a locally compact space Z , with spectral family $\mathcal{F} = \{m_{x,x'}^P\}_{x \in E, x' \in E'}$ whose range contains the range of every P_n . If the sequence $\{f_n\}$ constructed as in Theorem 2 converges to some f in $m_{x,x'}^P$ -measure for every x, x' , then there is an element $S = \int_Z f dP$ in $L(E, E)$ such that $\langle S_n x, x' \rangle \rightarrow \langle Sx, x' \rangle$ for every x, x' . If also each measure $f \cdot m_{x,x'}^P$ has compact support, then S is scalar.

In itself this rather complicated corollary does not say much more than the previous theorem. However it points to the connection with Foguel's results as follows. When E is a Banach space and S_n is a sequence converging strongly to an operator S then the sequence $\{S_n\}$ is bounded and all the measures involved will have compact support. From the convergence of S_n to S we can derive the existence of a function f to which the f_n converge in measure. All that is lacking is the spectral measure P . But this is derived from Theorem 4 below provided E is weakly complete and the B.A. generated by the ranges of the P_n is bounded.

It should be observed that these results may also be sharpened in ordered vector spaces. Such material may be found in [13], where Theorem 6 (p. 154) is clearly related to the above work.

4. Boolean algebras of projections. In this section the connection between Theorem 3 above and Foguel's work is completed. In the course of this development a number of theorems of Bade's are extended to locally convex spaces. A B.A. of projections will mean a subset \mathfrak{B} of $L(E, E)$ consisting of commuting idempotents, including I and 0 , which is a B.A. under the Boolean operations; $P \vee Q = P + Q - PQ$ and $P \wedge Q = PQ$.

DEFINITION 1. We will say that a locally convex space is weakly complete if whenever we have a sequence $\{x_n\}$ of elements of E for which $\lim_n \langle x_n, x' \rangle$ exists for every $x' \in E'$, then there is an element $x \in E$ such that $\lim_n \langle x_n, x' \rangle = \langle x, x' \rangle$ for all $x' \in E'$.

THEOREM 4. *Let E be weakly complete, and let \mathfrak{B} be a bounded B.A. of projections on E . Then there is a compact space Z and a spectral measure P on Z whose range contains \mathfrak{B} .*

Proof. Let Z be the Stone representation space of \mathfrak{B} , and let Σ be the field of all open-closed subsets of Z . Let $P_1: \Sigma \rightarrow \mathfrak{B}$ be the canonical isomorphism between Σ and \mathfrak{B} . For each $x \in E$, $x' \in E'$, define a set function $m_{x,x'}$ on Σ by the equation

$$m_{x,x'}(\sigma) = \langle P_1(\sigma)x, x' \rangle, \quad \sigma \in \Sigma.$$

It is immediate that each such $m_{x,x'}$ is finitely additive and bounded, using the properties of \mathfrak{B} . Since every set of Σ is both open and closed, $m_{x,x'}$ is regular. Hence, as in [5, Chapter III], there is a unique countably additive extension of each $m_{x,x'}$, which we denote by the same symbol, to the σ -field $S_0(Z)$ generated by Σ . If we can show that $\mathcal{F} = \{m_{x,x'}\}$ is a spectral family on Z , then its associated spectral measure P will have the properties we need.

We first show that \mathcal{F} is semispectral. Since $m_{\alpha x_1 + \beta x_2, x'}(\sigma) = \alpha m_{x_1, x'}(\sigma) + \beta m_{x_2, x'}(\sigma)$ and $m_{x, \gamma x'_1 + \delta x'_2}(\sigma) = \gamma m_{x, x'_1}(\sigma) + \delta m_{x, x'_2}(\sigma)$ for all $\sigma \in \Sigma$, and all scalars α, β, γ , and δ , it follows that the mapping $(x, x') \rightarrow m_{x,x'}$ is bilinear.

Next, using a technique taken from [10], we show that the set $\mathfrak{U}_1 = \{ \sum_{A \in \mathcal{D}} \alpha_A P_1(A) : \mathcal{D} \text{ a finite partition of } Z \text{ by sets of } \Sigma, \sup_{A \in \mathcal{D}} |\alpha_A| \leq 1 \}$ is equicontinuous. To see this, we note that for V a convex, balanced, closed neighborhood of zero in E , there is a family $\{x'_j\}_{j \in J}$ such that

$$V = \bigcap_{j \in J} \{x : |\langle x, x'_j \rangle| \leq 1\}.$$

Since E is barrelled and \mathfrak{B} is bounded, the set $\{P_1(\sigma) : \sigma \in \Sigma\}$ is equicontinuous. Therefore there is a neighborhood U of zero such that $x \in U$ implies $P_1(\sigma)x \in V$ for all $\sigma \in \Sigma$. Now, given an element of \mathfrak{U}_1 , let $\mathcal{D}' = \{A \in \mathcal{D} : \operatorname{Re} \langle P_1(A)x, x'_j \rangle \geq 0\}$. Then, for each $x \in U$, and $j \in J$,

$$\left| \sum_{A \in \mathcal{D}'} \alpha_A \operatorname{Re} \langle P_1(A)x, x'_j \rangle \right| \leq \operatorname{Re} \left\langle P_1 \left(\bigcup_{A \in \mathcal{D}'} A \right) x, x'_j \right\rangle \leq 1.$$

Treating negative real, positive imaginary, and negative imaginary parts in a similar fashion, we get $|\langle \sum_{A \in \mathcal{D}} \alpha_A P_1(A)x, x'_j \rangle| \leq 4$ for all $j \in J$ and $x \in U$. Thus $\sum_{A \in \mathcal{D}} \alpha_A P_1(A)x \in 4V$ if $x \in U$, and \mathfrak{U}_1 is equicontinuous.

Let \mathfrak{U} be the algebra of functions $f: Z \rightarrow C$ of the form $f(z) = \sum_{i=1}^n \alpha_i \phi_{\sigma_i}(z)$, with the α_i being scalars and $\{\sigma_i\}$ a finite disjoint partition of Z by sets of Σ . For each such f , we define $U_f \in L(E, E)$ by the equation $U_f = \sum_{i=1}^n \alpha_i P_1(\sigma_i)$. It then follows, using the equicontinuity of \mathfrak{U}_1 proved in the previous paragraph, that

$$B_x = \{U_f x : f \in \mathfrak{U}_1\} = \{U_f x : f \in \mathfrak{U}, \sup |f(z)| \leq 1\}$$

is bounded for each x in E . Also, for each equicontinuous set A in E' , the set

$$(28) \quad \{ 'U_f x' : f \in \mathfrak{A}, \sup |f(z)| \leq 1; x' \in A \}$$

is an equicontinuous subset of E' .

We observe next that \mathfrak{A} is dense in $\mathcal{K}(Z)$. To see this we note first that since the sets σ_i are open-closed, every f in \mathfrak{A} is continuous. It is obvious then that \mathfrak{A} is a subalgebra of $\mathcal{K}(Z)$, contains the identity, and contains \bar{f} whenever it contains f . Finally, since the sets of Σ contain a base for the topology of Z , the functions in \mathfrak{A} separate points, and by the Stone-Weierstrass Theorem, $\bar{\mathfrak{A}} = \mathcal{K}(Z)$. From this it follows that for each $x \in E$, $x' \in E'$,

$$(29) \quad \|m_{x,x'}\| = \sup_{f \in \mathfrak{A}_1} |\langle U_f x, x' \rangle|.$$

Using (27), for any $\varepsilon > 0$, let $V_{x,\varepsilon}$ be the neighborhood of zero in E' defined by

$$V_{x,\varepsilon} = \{x' \in E' : |\langle y, x' \rangle| \leq \varepsilon \text{ for all } y \in B_x\}.$$

Hence, from (29), $\|m_{x,x'}\| \leq \varepsilon$ for all $x' \in V_{x,\varepsilon}$, which shows that the mapping from E' to $\mathfrak{M}^1(Z)$ defined by $(x, x') \rightarrow m_{x,x'}$ is continuous for each fixed x .

Similarly, from (28), we see that for each $\varepsilon > 0$ and each equicontinuous set A in E' , there is a neighborhood of zero in E , $V_{A,\varepsilon}$, such that

$$|\langle y, 'U_f x' \rangle| \leq \varepsilon \quad \text{for all } f \in \mathfrak{A}_1, x' \in A.$$

Since A is arbitrary, it follows, again using (18), that the mapping $(x, x') \rightarrow m_{x,x'}$ is \mathcal{E} -hypocontinuous, and, by letting $A = \{x'\}$, continuous from E to $\mathfrak{M}^1(Z)$ for each x' . Hence the family \mathcal{F} is semispectral.

Since $B^\infty(Z) \subseteq L^\infty(\mathcal{F})$, it follows from remarks in the section on preliminary concepts that whenever one has a semispectral family \mathcal{F} , then for every $f \in B^\infty(Z)$, there is a linear mapping, which we denote by \hat{U}_f , from E to E'' satisfying $\langle \hat{U}_f x, x' \rangle = \int_Z f dm_{x,x'}$ for every $x \in E$, $x' \in E'$. Moreover, this mapping is continuous if E'' is given the topology of uniform convergence on sets of \mathcal{E} . If we denote by $x \rightarrow \hat{x}$ the canonical embedding of E in E'' , it is evident that for each $f \in \mathfrak{A}$, and each $x' \in E'$, $\langle \hat{U}_f x, x' \rangle = \langle [U_f x]^\wedge, x' \rangle$. Since \mathfrak{A} is dense in $\mathcal{K}(Z)$, for each $f \in B_0^\infty(Z)$, there is a sequence $\{f_n\} \subset \mathfrak{A}$ such that

$$\lim_n m_{x,x'}(f_n) = m_{x,x'}(f).$$

Hence $\lim_n \langle U_{f_n} x, x' \rangle$ exists for every $x \in E$, $x' \in E'$. Since E is weakly complete, there is an element of E (necessarily unique and well defined independent of the choice of the sequence $f_n \rightarrow f$, by virtue of the next equation) which we denote $U_f x$, such that

$$\lim_n \langle U_{f_n} x, x' \rangle = \langle U_f x, x' \rangle.$$

Hence $[U_f x]^\wedge = \hat{U}_f x$ for every $x \in E$, $f \in B_0^\infty(Z)$. Since polars of sets in \mathcal{E} generate the topology of E , $U_f \in L(E, E)$.

Finally, for every f and $g \in \mathfrak{U}$, supposing

$$f = \sum_{i=1}^n \alpha_i \phi_{\sigma_i} \text{ and } g = \sum_{j=1}^m \beta_j \phi_{\sigma_j},$$

then

$$\begin{aligned} U_f U_g x &= \sum_{i=1}^n \alpha_i P(\sigma_i) \sum_{j=1}^m \beta_j P(\sigma_j) x \\ &= \sum_{i,j=1}^{n,m} \alpha_i \beta_j P(\sigma_i \cap \sigma_j) x = U_{fg} x. \end{aligned}$$

Again using the fact that \mathfrak{U} is dense in $\mathcal{K}(Z)$ we find that $U_f U_g = U_{fg}$ for every f and $g \in B_0^\infty(Z)$. This completes the proof that \mathcal{F} is spectral.

Following Bade [2], we give the following:

DEFINITION 2. A B.A. \mathfrak{B} of projections on E is said to be complete (σ -complete) if for each subset (subsequence) $\{P_\alpha\} \subset \mathfrak{B}$, E may be decomposed into the (topological) direct sum $E = \mathfrak{M} \oplus \mathfrak{N}$, where

$$\mathfrak{M} = \text{clm}^{(2)} \{P_\alpha(E)\}, \quad \mathfrak{N} = \bigcap_\alpha (I - P_\alpha)E,$$

and the projection P_0 which maps E onto \mathfrak{M} and \mathfrak{N} into 0, is in \mathfrak{B} .

THEOREM 5. Let \mathfrak{B} be a bounded complete B.A. of projections on E , and $\{P_\alpha\}$ an increasing (decreasing) net in \mathfrak{B} . Then

$$\lim_\alpha P_\alpha x = \left(\bigvee_\alpha P_\alpha \right) x \quad \text{for all } x \in E.$$

If \mathfrak{B} is σ -complete and bounded, corresponding results are true for sequences.

Proof. Let $\{P_\alpha\}$ be increasing, $P_0 = \bigvee_\alpha P_\alpha$. Fix $x \in E$ and let V be a neighborhood of zero in E . Find U , a neighborhood of zero in E , such that $U + U \subset V$. Since \mathfrak{B} is bounded and E is barreled, \mathfrak{B} is equicontinuous and there is W , a neighborhood of zero in E , such that $P(W) \subset U$ for all $P \in \mathfrak{B}$. Since $P_0(E) = \text{clm} \{P_\alpha(E)\}$, we can find $y = \sum_{i=1}^n \beta_i z_i$ such that $y - P_0 x \in W \cap U$, and each $z_i \in P_{\alpha_i}(E)$ for some α_i . Pick α_0 so that $\alpha_0 \geq \alpha_i$ for $i = 1, 2, \dots, n$. Then for $\alpha \geq \alpha_0$, $P_\alpha y = y$ so that

$$\begin{aligned} P_\alpha x - P_0 x &= (P_\alpha x - y) + (y - P_0 x) \in P_\alpha(P_0 x - y) + U \\ &\subseteq U + U \subseteq V. \end{aligned}$$

Since V is arbitrary, $\lim_\alpha P_\alpha x = P_0 x$. The proof for the decreasing case and for sequences is similar.

This theorem may also be considered to be a corollary of the (relatively easy extension to locally convex spaces of a) theorem of Barry's [3].

(2) clm = smallest closed linear manifold containing $\{ \}$.

THEOREM 6. *Let \mathfrak{B} be a B.A. of projections in E . If \mathfrak{B} is bounded and σ -complete, then it is the range of a spectral measure on its Stone representation space Z .*

Proof. As usual, let P be the isomorphism of the field Σ of open-closed sets of Z onto \mathfrak{B} . For each $\sigma \in S_0(Z)$ there is $\sigma' \in \Sigma$ such that the symmetric difference, $\sigma \Delta \sigma'$, is a set of Baire Cat.I. If σ'' is another set of Σ such that $\sigma \Delta \sigma''$ is Cat.I., then so is $\sigma' \Delta \sigma''$ of Cat.I. Since both σ' and σ'' are in Σ , this means that $\sigma \Delta \sigma''$ is open, which is impossible unless $\sigma' = \sigma''$. Hence there is no inconsistency in defining P_1 for σ in $S_0(Z)$ by $P_1(\sigma) = P(\sigma')$. The only point at issue is countable additivity. Let $\{\sigma_n\}$ be a disjoint sequence of sets in $S_0(Z)$, $\{\sigma'_n\}$ the corresponding sets in Σ . Using the σ -completeness of \mathfrak{B} there is a set $\sigma' \in \Sigma$ such that $P(\sigma') = \bigvee_n P(\sigma'_n)$. Then $\sigma' \Delta (\bigcup_n \sigma'_n)$ is Cat.I. so that $P_1(\bigcup_n \sigma_n) = P(\sigma')$. Finally, from the previous theorem, $(\bigvee_n P(\sigma'_n))x = \sum_n (P(\sigma'_n)x) = \sum_n (P_1(\sigma_n)x)$ for each $x \in E$.

THEOREM 7. *A convex subset of $L(E, E)$ has the same closure in the topology of convergence at each point of E as it does in the topology of convergence for each pair $x \in E, x' \in E'$. (I.e. in what are referred to as strong and weak operator topologies in the theory of Banach spaces.)*

Proof. This is just a repetition of Lemma 3.3 in [1]. The proof given there may be translated into the language of locally convex spaces without serious difficulty, the specific properties of norms being irrelevant here. As an illustration of how the translation may proceed, at one point a Banach space of n -tuples with maximum norm is constructed. If the topology of E is generated by seminorms $\{p_\alpha\}$, each such p_α may be used to generate a seminorm \hat{p}_α on the space of n -tuples $\hat{x} = [x_1, \dots, x_n]$ of elements of E by letting $\hat{p}_\alpha(\hat{x}) = \max_{1 \leq i \leq n} p_\alpha(x_i)$. These seminorms then generate a locally convex topology on $E \times E \times \dots \times E$ (n factors).

THEOREM 8. *Let \mathfrak{B} be a bounded complete B.A. of projections on E . Let \mathfrak{A} be the algebra generated by \mathfrak{B} in $L(E, E)$, $\overline{\mathfrak{A}}^w$ ($= \overline{\mathfrak{A}}^s$ by the previous theorem), its closure in the weak operator topology. If $T \in \overline{\mathfrak{A}}^w$ and if K and K' are finite subsets of E such that $T(K') = 0$, then there is $P_0 = P_{K, K'} \in \mathfrak{B}$ such that $T(K) \subseteq P_0(E)$ and $K' \subseteq (I - P_0)(E)$.*

Proof. The theorem asserts the existence of a $P_0 \in \mathfrak{B}$ such that $PTy = Ty$ and $Pz = 0$ for all $y \in K, z \in K'$. Since $T \in \overline{\mathfrak{A}}^s$, for any neighborhood V of zero in E there is $A_V \in \mathfrak{A}$ such that $A_V y - Ty \in V$ and $A_V z \in V$ for all $y \in K, z \in K'$. The cardinality of K and K' is irrelevant (though it must be finite) from this point on. To simplify the computations we suppose that K and K' each contain a single point, $K = \{y\}$ and $K' = \{z\}$, and that \mathfrak{D} is a filter base of neighborhoods of zero in E consisting in closed convex balanced sets. The proof is modeled on that

of Theorem 3.4 in [1]. However, the "translation" is fairly involved and is given with some completeness.

For each $V \in \mathfrak{D}$, $A_V = \sum_{i=1}^n \alpha_{i,V} P(\sigma_{i,V})$ where P is the spectral measure whose range is \mathfrak{B} , and the $\sigma_{i,V}$ are clopen subsets of the Stone space of \mathfrak{B} . For each $\varepsilon > 0$, $V \in \mathfrak{D}$, let $\sigma_V^\varepsilon = \bigcup \{\sigma_{i,V} : |\alpha_{i,V}| > \varepsilon\}$. Obviously, if $\varepsilon_1 \leq \varepsilon_2$, then $P(\sigma_V^{\varepsilon_1}) \leq P(\sigma_V^{\varepsilon_2})$. Now define

$$P_{V,\varepsilon} = \bigvee \{P(\sigma_W^\varepsilon) : W \in \mathfrak{D}, W \subseteq V\}.$$

Then $\varepsilon_1 \leq \varepsilon_2$ implies

$$(30) \quad P_{V,\varepsilon_2} \leq P_{V,\varepsilon_1}.$$

Likewise, for each ε , if $V_1 \subseteq V_2$, both elements of \mathfrak{D} , then $P_{V_1,\varepsilon} \leq P_{V_2,\varepsilon}$. Consequently if $V_1 \subseteq V_2$ and $\varepsilon_1 \leq \varepsilon_2$, then $P_{V_1,\varepsilon_2} P_{V_2,\varepsilon_1} = P_{V_1,\varepsilon_2}$. More important, for each ε ,

$$P_\varepsilon x = (\bigwedge P_{V,\varepsilon})x = \lim_V P_{V,\varepsilon} x \quad \text{for all } x \in E.$$

From (30) we see that $\varepsilon_1 \leq \varepsilon_2$ implies $P_{\varepsilon_1} \geq P_{\varepsilon_2}$. Hence $P_0 x = (\bigvee_{\varepsilon > 0} P_\varepsilon)x = \lim_{\varepsilon \rightarrow 0} P_\varepsilon x$, $x \in E$. Now

$$(31) \quad P_{V,\varepsilon} T y - T y = P_{V,\varepsilon} (T y - A_V y) + (P_{V,\varepsilon} A_V y - A_V y) + (A_V y - T y).$$

Thus given $W \in \mathfrak{D}$, we find $U \in \mathfrak{D}$ such that $U + U + U \subseteq W$ and $\beta_U > 0$ such that $\|f\| \leq 1$ implies $U_f y \in \beta_U U$. Since $P_{V,\varepsilon} \geq P(\sigma_V^\varepsilon)$ for all V and ε , $P_{V,\varepsilon} A_V - A_V = U_y$ for some y satisfying $\|y\| \leq \varepsilon$. Hence $(P_{V,\varepsilon} A_V - A_V)y \in U$ for all V , all $\varepsilon \leq 1/\beta_U$. By the equicontinuity of \mathfrak{B} we may find V' such that $P_{V,\varepsilon}(T y - A_V y) \in U$ and $A_V y - T y \in V \subseteq U$ for all ε and all $V \subseteq V'$.

Substituting these results in (31) we find that given W , there exists $V' \in \mathfrak{D}$ and $\gamma > 0$ ($\gamma = 1/\beta_U$) such that

$$P_{V,\varepsilon} T y - T y \in W \quad \text{for all } V \subseteq V', \varepsilon \leq \gamma.$$

It follows readily that

$$P_\varepsilon T y - T y \in W \quad \text{for all } \varepsilon \leq \gamma.$$

Hence $P_0 T y - T y \in W$. Since W is arbitrary $P_0 T y = T y$.

Finally, to show $P_0 z = 0$, we define

$$A'_V = \sum \beta_{i,V} P(\sigma_{i,V})$$

where $\beta_{i,V} = 1/\alpha_{i,V}$ whenever $\alpha_{i,V} \neq 0$, and $\beta_{i,V} = 0$ otherwise. Then

$$P(\sigma_V^\varepsilon) A'_V = \sum_{|\beta_{i,V}| < 1/\varepsilon} \beta_{i,V} P(\sigma_{i,V}).$$

Again using the equicontinuity of $\{U_f : \|f\| \leq 1\}$, given $U \in \mathfrak{D}$ and $\varepsilon > 0$ we may find V such that

$$(32) \quad P_{V,\delta} z \in U \quad \text{for all } \delta \geq \varepsilon.$$

Using the equicontinuity of \mathfrak{B} we may assume that there is a W such that (32) holds for all $V \supseteq W$ so that $P_\delta z \in U$ for all $\delta \geq \varepsilon$. Hence $P_0 z \in U$. Since U is arbitrary $P_0 z = 0$.

This theorem is really just a lemma leading to the next theorem.

THEOREM 9. *A bounded complete B.A. of projections contains every projection in the weakly closed algebra it generates.*

Proof. As in the first paragraph of the proof of Theorem 3.4 of [1]. The hard part is done above.

THEOREM 10. *If \mathfrak{B} is a bounded complete B.A. of projections then $\mathfrak{B} = \overline{\mathfrak{B}}^s$.*

Proof. $T \in \overline{\mathfrak{B}}^s$ implies $T \in \overline{\mathfrak{A}}^w$ and $T^2 = T$. Hence $T \in \mathfrak{B}$ by the previous theorem.

The last theorem given here is a generalization of Theorem 2.7 in [2]. We are indebted to Bertram Walsh for a simplification of the hypotheses.

THEOREM 11. *Let E be metrizable, \mathfrak{B} a bounded σ -complete B.A. of projections on E . Then $\overline{\mathfrak{B}}^s$ is complete.*

Proof. Again the proof is in outline a translation of that given in [2]. The condition of metrizability on E is used to accomplish countable decomposability for the restriction \mathfrak{B}_x of \mathfrak{B} to the cyclic subspace

$$\mathfrak{M}(x) = \text{clm} \{Px : P \in \mathfrak{B}\}.$$

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